Simplifying Kaufman's Solution of the Two-Dimensional Ising Model

Boris Kastening*
Institut für Theoretische Physik
Freie Universität Berlin
Arnimallee 14
D-14195 Berlin
Germany
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We considerably simplify Kaufman's solution of the two-dimensional Ising model by introducing two commuting representations of the complex rotation group SO$(2n, C)$. All eigenvalues of the transfer matrix and therefore the partition function are found in a straightforward way.

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Since Onsager's solution [1] in the transfer matrix approach [2] of the two-dimensional Ising model [3] with vanishing magnetic field and its subsequent simplification by Kaufman [4], there have been a number of related as well as alternative solutions, see e.g. Baxter's book on exactly solved models in statistical mechanics [5] and references therein. Among the transfer matrix solutions are the ones by Schultz, Mattis and Lieb, by Thompson, by Baxter and by Stephen and Mittag [6]. Nevertheless, the author of this work feels that there is still room for a nice and straightforward solution. A completely self-contained and detailed account of this work may be found in [7].

We study the two-dimensional Ising model with zero magnetic field on a square lattice with m rows and n columns subject to toroidal boundary conditions. The transfer matrix is expressed in terms of the generators of two commuting representations of the complex rotation group SO$(2n, C)$. These representations naturally arise from projected bilinears of $2^n \times 2^n$ spin matrices. Conservatively speaking, we reduce Kaufman's approach to its essential steps, avoiding in particular the doubling of the number of eigenvalues of the transfer matrix and subsequent rather involved arguments for the choice of the correct ones. Additionally, there is no need to investigate the transformation properties of the spin matrices.

Our notation is in the spirit of [8], with sans serif capitals reserved for $2^n \times 2^n$ matrices. The structure of this work is as follows: After defining the model and its transfer matrix T, we express T in terms of $2^n \times 2^n$ spin matrices X_ν, Y_ν, Z_ν. A rescaled transfer matrix V is defined whose eigenvalues are, up to a trivial factor, the eigenvalues of T. We define further spin matrices Γ_ν and two commuting projection classes $\mathsf{J}^+_{\alpha\beta}$ and $\mathsf{J}^-_{\alpha\beta}$ of their bilinears. After investigating the relevant properties of the $\mathsf{J}^\pm_{\alpha\beta}$, we express V in terms of them. We introduce $2n \times 2n$ matrices $J_{\alpha\beta}$ whose algebra is identical to that of $\mathsf{J}^\pm_{\alpha\beta}$ and define matrices $V^\pm$ in terms of the $J_{\alpha\beta}$ such that the relation between $V^\pm$ and $J_{\alpha\beta}$ is closely related to that between V and $\mathsf{J}^\pm_{\alpha\beta}$. The result of the well-known diagonalization procedure for the $V^\pm$ is given and the analogy between $V^\pm$ and V exploited for the diagonalization of V. The eigenvalues of V and the partition function are found explicitly.

The energy is given by $E = E_a + E_b$ with

$$E_a = J_a \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} s_{\mu\nu} s_{\mu+1,\nu}, \qquad E_b = J_b \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} s_{\mu\nu} s_{\mu,\nu+1}, \tag{1}$$

with $\beta^{-1} = kT$. J_a and J_b are temperature-independent interaction energy parameters that we assume to be negative. We identify rows 1 and m+1 and columns 1 and n+1, i.e., the lattice is wrapped on a torus. The $s_{\mu\nu}$ can take the values ± 1. The partition function is then given by

$$Z(a, b) = \sum_{s_{11}} \cdots \sum_{s_{mn}} \exp(-\beta E), \tag{2}$$

with the definitions $a = -\beta J_a$ and $b = -\beta J_b$. Z can be expressed with the help of a $2^n \times 2^n$ transfer matrix T, $Z(a, b) = \mathrm{Tr}\, \mathsf{T}^m$ with T defined by its elements ($s_{n+1} \equiv s_1$),

$$\langle \pi | \mathsf{T} | \pi' \rangle = \prod_{\nu=1}^{n} \exp(a s_\nu s'_\nu + b s_\nu s_{\nu+1}), \tag{3}$$

where $\pi_\mu = \{s_{\mu 1}, \ldots, s_{\mu n}\}$ for $\mu = 1, \ldots, m$. We can split T into a product of two matrices $\mathsf{T} = \mathsf{V}_b \mathsf{V}'_a$, defining V'_a and V_b by their elements

$$\langle \pi | \mathsf{V}'_a | \pi' \rangle \;=\; \prod_{\nu=1}^{n} \exp(a s_\nu s'_\nu), \tag{4}$$

$$\langle \pi | \mathsf{V}_b | \pi' \rangle \;=\; \prod_{\nu=1}^{n} \delta_{s_\nu s'_\nu} \exp(b s_\nu s_{\nu+1}). \tag{5}$$

With the help of the Pauli matrices σ_x, σ_y, σ_z, and the 2×2 unit matrix $\mathbf{1}$, we define Hermitian $2^n \times 2^n$ spin matrices by the direct products

$$\mathsf{X}_\nu = \left(\bigotimes_{\nu'=1}^{\nu-1} \mathbf{1} \right) \otimes \sigma_x \otimes \left(\bigotimes_{\nu''=\nu+1}^{n} \mathbf{1} \right), \tag{6}$$

*Email address: ka@physik.fu-berlin.de

and analogously for Y_ν and Z_ν in terms of σ_y and σ_z, respectively. With $\bar{a} > 0$ defined by $\sinh(2\bar{a})\sinh(2a) = 1$, we can write $\mathsf{V}'_a = [2\sinh(2a)]^{n/2}\mathsf{V}_a$ with

$$\mathsf{V}_a = \prod_{\nu=1}^{n} \exp\left(\bar{a}\mathsf{X}_\nu\right), \tag{7}$$

and

$$\mathsf{V}_b = \prod_{\nu=1}^{n} \exp\left(b\mathsf{Z}_\nu\mathsf{Z}_{\nu+1}\right), \tag{8}$$

where we have identified $\mathsf{Z}_{n+1} = \mathsf{Z}_1$. The transfer matrix may then be expressed as $\mathsf{T} = [2\sinh(2a)]^{n/2}\mathsf{V}_b\mathsf{V}_a$. Due to the cyclic property of the trace, we may rewrite the partition function (2) as

$$Z(a,b) = [2\sinh(2a)]^{mn/2}\mathrm{Tr}\mathsf{V}^m, \tag{9}$$

where V is defined by the Hermitian matrix

$$\mathsf{V} = \mathsf{V}_{a/2}\mathsf{V}_b\mathsf{V}_{a/2} \tag{10}$$

with

$$\mathsf{V}_{a/2} = \prod_{\nu=1}^{n} \exp(\bar{a}\mathsf{X}_\nu/2) \tag{11}$$

so that $\mathsf{V}_{a/2}^2 = \mathsf{V}_a$. If Λ_k are the 2^n eigenvalues of V, we have

$$Z(a,b) = [2\sinh(2a)]^{mn/2}\sum_{k=1}^{2^n}\Lambda_k^m. \tag{12}$$

Our task is therefore to find the eigenvalues of V.

Define the $2n$ matrices $(\nu = 1,\ldots,n)$

$$\begin{aligned} \Gamma_{2\nu-1} &= \mathsf{X}_1\cdots\mathsf{X}_{\nu-1}\mathsf{Z}_\nu, &(13)\\ \Gamma_{2\nu} &= \mathsf{X}_1\cdots\mathsf{X}_{\nu-1}\mathsf{Y}_\nu, &(14)\end{aligned}$$

which obey

$$\{\Gamma_\mu,\Gamma_\nu\} = 2\delta_{\mu\nu}. \tag{15}$$

Define further the matrix

$$\mathsf{U}_\mathsf{X} = \mathsf{X}_1\cdots\mathsf{X}_n = i^n\Gamma_1\Gamma_2\cdots\Gamma_{2n}, \qquad \mathsf{U}_\mathsf{X}^2 = \mathbf{1}, \tag{16}$$

which anticommutes with every Γ_μ, $\{\Gamma_\mu,\mathsf{U}_\mathsf{X}\} = 0$. Now we can write for the matrices appearing in the exponents of Eqs. (8) and (11)

$$\begin{aligned} \mathsf{X}_\nu &= -\frac{i}{2}[\Gamma_{2\nu},\Gamma_{2\nu-1}], \quad \nu = 1,\ldots,n, &(17)\\ \mathsf{Z}_\nu\mathsf{Z}_{\nu+1} &= -\frac{i}{2}[\Gamma_{2\nu+1},\Gamma_{2\nu}], \quad \nu = 1,\ldots,n-1, &(18)\\ \mathsf{Z}_n\mathsf{Z}_1 &= \frac{i}{2}\mathsf{U}_\mathsf{X}[\Gamma_1,\Gamma_{2n}]. &(19)\end{aligned}$$

So far our treatment has been rather similar to Huang's write up [8] of Kaufman's approach [4]. Our subsequent treatment rests on the observation that the formulation (10) of V with Eqs. (8) and (11) involves only the product U_X of all Γ_ν and bilinears $\Gamma_\alpha\Gamma_\beta$, see Eqs. (16)–(19). This will allow us to express V in terms of the elements of two commuting algebras of projected bilinears of the Γ_ν.

With the help of the projectors

$$\mathsf{P}^\pm \equiv \frac{1}{2}(\mathbf{1} \pm \mathsf{U}_\mathsf{X}), \tag{20}$$

define the matrices

$$\mathsf{J}_{\alpha\beta} = -\frac{i}{4}[\Gamma_\alpha,\Gamma_\beta], \qquad \mathsf{J}^\pm_{\alpha\beta} = \mathsf{P}^\pm\mathsf{J}_{\alpha\beta}, \tag{21}$$

so that

$$\mathsf{J}_{\alpha\beta} = \mathsf{J}^+_{\alpha\beta} + \mathsf{J}^-_{\alpha\beta}, \qquad \mathsf{U}_\mathsf{X}\mathsf{J}^\pm_{\alpha\beta} = \pm\mathsf{J}^\pm_{\alpha\beta}. \tag{22}$$

Since $\mathsf{J}^\pm_{\alpha\beta} = -\mathsf{J}^\pm_{\beta\alpha}$, there are $n(2n-1)$ such independent matrices of each kind $\mathsf{J}^+_{\alpha\beta}$ and $\mathsf{J}^-_{\alpha\beta}$. It is straightforward to show that their algebra decomposes into two commuting parts, $[\mathsf{J}^+_{\alpha\beta},\mathsf{J}^-_{\gamma\delta}] = 0$, which obey identical algebras

$$[\mathsf{J}^\pm_{\alpha\beta},\mathsf{J}^\pm_{\gamma\delta}] = i(\delta_{\alpha\gamma}\mathsf{J}^\pm_{\beta\delta} + \delta_{\beta\delta}\mathsf{J}^\pm_{\alpha\gamma} - \delta_{\alpha\delta}\mathsf{J}^\pm_{\beta\gamma} - \delta_{\beta\gamma}\mathsf{J}^\pm_{\alpha\delta}). \tag{23}$$

Next note that with Eqs. (17)–(19), (21), and (22) we can write

$$\begin{aligned} \mathsf{X}_\nu &= 2(\mathsf{J}^+_{2\nu,2\nu-1}+\mathsf{J}^-_{2\nu,2\nu-1}), \quad \nu = 1,\ldots,n, &(24)\\ \mathsf{Z}_\nu\mathsf{Z}_{\nu+1} &= 2(\mathsf{J}^+_{2\nu+1,2\nu}+\mathsf{J}^-_{2\nu+1,2\nu}), \quad \nu = 1,\ldots,n-1, &(25)\\ \mathsf{Z}_n\mathsf{Z}_1 &= -2\mathsf{U}_\mathsf{X}(\mathsf{J}^+_{1,2n}+\mathsf{J}^-_{1,2n}) = -2(\mathsf{J}^+_{1,2n}-\mathsf{J}^-_{1,2n}). &(26)\end{aligned}$$

This allows us to express $\mathsf{V}_{a/2}$ from Eq. (11) and V_b from Eq. (8) in terms of the $\mathsf{J}^\pm_{\alpha\beta}$,

$$\mathsf{V}_{a/2} = \prod_{\nu=1}^{n}\exp[\bar{a}(\mathsf{J}^+_{2\nu,2\nu-1} + \mathsf{J}^-_{2\nu,2\nu-1})] = \mathsf{V}^+_{a/2}\mathsf{V}^-_{a/2} \tag{27}$$

with

$$\mathsf{V}^\pm_{a/2} = \prod_{\nu=1}^{n}\exp(\bar{a}\mathsf{J}^\pm_{2\nu,2\nu-1}), \tag{28}$$

and

$$\begin{aligned} \mathsf{V}_b &= \exp[-2b(\mathsf{J}^+_{1,2n}-\mathsf{J}^-_{1,2n})]\\ &\quad\times\prod_{\nu=1}^{n-1}\exp[2b(\mathsf{J}^+_{2\nu+1,2\nu}+\mathsf{J}^-_{2\nu+1,2\nu})]\\ &= \mathsf{V}^+_b\mathsf{V}^-_b &(29)\end{aligned}$$

with

$$\mathsf{V}^\pm_b = \exp(\mp 2b\mathsf{J}^\pm_{1,2n})\prod_{\nu=1}^{n-1}\exp(2b\mathsf{J}^\pm_{2\nu+1,2\nu}). \tag{30}$$

The rescaled transfer matrix V defined in Eq. (10) reads then $\mathsf{V} = \mathsf{V}^+\mathsf{V}^-$ with

$$\mathsf{V}^\pm = \mathsf{V}^\pm_{a/2}\mathsf{V}^\pm_b\mathsf{V}^\pm_{a/2}, \qquad [\mathsf{V}^+,\mathsf{V}^-] = 0. \tag{31}$$

Define $N \times N$ matrices $J_{\alpha\beta}$ by their elements

$$(J_{\alpha\beta})_{ij} = -i(\delta_{\alpha i}\delta_{\beta j} - \delta_{\beta i}\delta_{\alpha j}), \tag{32}$$

where Greek and Latin indices run from 1 to N. Since $J_{\alpha\beta} = -J_{\beta\alpha}$, there are $N(N-1)/2$ such independent matrices. As can be easily checked, they obey the algebra (23), if we set $N = 2n$. Now consider the matrices $S = \exp(ic_{\alpha\beta}J_{\alpha\beta})$, where $c_{\alpha\beta}$ are arbitrary complex numbers. The matrices S form the group $\mathrm{SO}(N, C)$ of complex $N \times N$ matrices with $S^T = S^{-1}$, $\det S = 1$.

Define the $\mathrm{SO}(2n, C)$ matrices

$$V^{\pm} = V_{a/2} V_b^{\pm} V_{a/2} \tag{33}$$

with

$$V_{a/2} = \prod_{\nu=1}^{n} \exp(\bar{a} J_{2\nu,2\nu-1}) \tag{34}$$

and

$$V_b^{\pm} = \exp(\mp 2b J_{1,2n}) \prod_{\nu=1}^{n-1} \exp(2b J_{2\nu+1,2\nu}), \tag{35}$$

in analogy with $\mathsf{V}^{\pm}$, $\mathsf{V}^{\pm}_{a/2}$, and $\mathsf{V}^{\pm}_b$ in Eqs. (31), (28), and (30). Since $\bar{a}$ and b are real, the matrices $V_{a/2}$, $V_b^{\pm}$, and $V^{\pm}$ are not only orthogonal, but also Hermitian, so the $V^{\pm}$ have only real eigenvalues and in each case a complete set of orthonormal eigenvectors.

It is well known [4, 8] how to diagonalize matrices of the types $V^{\pm}$. Applying similarity transformations $V_S^{\pm} \equiv S_{\pm} V^{\pm} S_{\pm}^{-1}$ with certain explicitly known matrices $S_{\pm}$, one obtains

$$V_S^{\pm} = \exp\left(\sum_{\nu=1}^{n} \gamma_{\{{2\nu-1 \atop 2\nu-2}\}} J_{2\nu,2\nu-1} \right) \tag{36}$$

with γ_k defined by

$$\cosh \gamma_k = \cosh 2\bar{a} \cosh 2b - \cos \frac{\pi k}{n} \sinh 2\bar{a} \sinh 2b. \tag{37}$$

We fix the sign of γ_k by defining $\gamma_k = 2\bar{a}$ for $b = 0$ and then analytically continuing to other values of b. For $k = 1, \ldots, 2n-1$, this means $\gamma_k > 0$. On the other hand, for γ_0 this means

$$\gamma_0 = 2(\bar{a} - b). \tag{38}$$

Our sign convention for the γ_k and in particular for γ_0 allows us to treat all γ_k on an equal footing for both $\bar{a} > b$ and $\bar{a} < b$, i.e., irrespective of the temperature. Note that the $V_S^{\pm}$ are only block diagonal with 2×2 blocks. It would be trivial to diagonalize $\mathsf{V}_S^{\pm}$, but the form given by Eq. (36) is most convenient for our purposes.

It is straightforward to show that the matrices $S_{\pm}$ are elements of $\mathrm{SO}(N, C)$ and may therefore be written as $S_{\pm} = \exp(ic^{\pm}_{\alpha\beta} J_{\alpha\beta})$. Now use the same parameters $c^{\pm}_{\alpha\beta}$ to define the $2^n \times 2^n$-dimensional transformation matrix

$$\mathsf{S} = \mathsf{S}_+ \mathsf{S}_-, \qquad \mathsf{S}_{\pm} = \exp(ic^{\pm}_{\alpha\beta} \mathsf{J}^{\pm}_{\alpha\beta}), \tag{39}$$

and write

$$\mathsf{V}_\mathsf{S} = \mathsf{S}\mathsf{V}\mathsf{S}^{-1} = \mathsf{S}_+ \mathsf{V}^+ \mathsf{S}_+^{-1} \mathsf{S}_- \mathsf{V}^- \mathsf{S}_-^{-1} \equiv \mathsf{V}_S^+ \mathsf{V}_S^-. \tag{40}$$

The factors defining $\mathsf{V}_S^{\pm}$ in terms of $\bar{a}$, b, $c^{\pm}_{\alpha\beta}$, and the $\mathsf{J}^{\pm}_{\alpha\beta}$ have the same structure as the $V_S^{\pm}$ in terms $\bar{a}$, b, $c^{\pm}_{\alpha\beta}$, and the $J_{\alpha\beta}$. Now imagine using the Baker-Campbell-Hausdorff formula [9]

$$\exp(A)\exp(B) = \exp(A+B-\tfrac{1}{2}[B,A]+\tfrac{1}{12}\{[A,[A,B]]+[B,[B,A]]\}+\cdots) \tag{41}$$

to work out all products of exponentials in $\mathsf{V}_S^{\pm}$. Since the $\mathsf{J}^+_{\alpha\beta}$, $\mathsf{J}^-_{\alpha\beta}$, and $J_{\alpha\beta}$ obey identical algebras, the result is

$$\mathsf{V}_\mathsf{S}^{\pm} = \exp\left(\sum_{\nu=1}^{n} \gamma_{\{{2\nu-1 \atop 2\nu-2}\}} \mathsf{J}^{\pm}_{2\nu,2\nu-1} \right), \tag{42}$$

so that

$$\begin{aligned} \mathsf{V}_\mathsf{S} &= \exp\left[\sum_{\nu=1}^{n} (\gamma_{2\nu-1} \mathsf{J}^+_{2\nu,2\nu-1} + \gamma_{2\nu-2} \mathsf{J}^-_{2\nu,2\nu-1}) \right] \\ &= \exp\left[\frac{1}{4} \sum_{\nu=1}^{n} \gamma_{2\nu-1} (\mathbf{1} + \mathsf{U}_\mathsf{X}) \mathsf{X}_\nu \right. \\ &\qquad \left. + \frac{1}{4} \sum_{\nu=1}^{n} \gamma_{2\nu-2} (\mathbf{1} - \mathsf{U}_\mathsf{X}) \mathsf{X}_\nu \right], \end{aligned} \tag{43}$$

with the same γ_k as defined in Eq. (37) and the subsequent sign convention.

To diagonalize V_S, define another similarity transformation $\mathsf{V}_\mathsf{Y} = \mathsf{R}_\mathsf{Y} \mathsf{V}_\mathsf{S} \mathsf{R}_\mathsf{Y}^{-1}$ with R_Y and its inverse given by

$$\mathsf{R}_\mathsf{Y}^{\pm 1} = 2^{-n/2} \prod_{\nu=1}^{n} (\mathbf{1} \pm i\mathsf{Y}_\nu). \tag{44}$$

Since $\mathsf{R}_\mathsf{Y} \mathsf{X}_\nu \mathsf{R}_\mathsf{Y}^{-1} = \mathsf{Z}_\nu$, this transformation takes V_S into

$$\begin{aligned} &\mathsf{V}_\mathsf{Y} = \mathsf{R}_\mathsf{Y} \mathsf{V}_\mathsf{S} \mathsf{R}_\mathsf{Y}^{-1} \\ &= \exp\left[\frac{1}{4} \sum_{\nu=1}^{n} \gamma_{2\nu-1} (\mathbf{1}+\mathsf{U}_\mathsf{Z}) \mathsf{Z}_\nu + \frac{1}{4} \sum_{\nu=1}^{n} \gamma_{2\nu-2} (\mathbf{1}-\mathsf{U}_\mathsf{Z}) \mathsf{Z}_\nu \right] \end{aligned} \tag{45}$$

with $\mathsf{U}_\mathsf{Z} = \mathsf{Z}_1 \cdots \mathsf{Z}_n$.

The matrix V_Y is diagonal, but we still have to determine its elements. U_Z is a diagonal matrix with elements $+1$ and -1 occurring in equal numbers. For each element holds, if an even (odd) number of Z_ν provides a factor -1, the matrix element of U_Z is $+1$ (-1). This means (i) a matrix element of $(\mathbf{1} + \mathsf{U}_\mathsf{Z})/2$ is 1 (0) if an even (odd)

number of Z_ν provides a factor -1, (ii) a matrix element of $(\mathbf{1}-\mathsf{U}_\mathsf{Z})/2$ is 1 (0) if an odd (even) number of Z_ν provides a factor -1. It follows that the 2^n eigenvalues of V split into 2^{n-1} eigenvalues of the form

$$\exp\left(\frac{1}{2}\sum_{\nu=1}^{n}(\pm)\gamma_{2\nu-1}\right), \tag{46}$$

and 2^{n-1} eigenvalues of the form

$$\exp\left(\frac{1}{2}\sum_{\nu=1}^{n}(\pm)\gamma_{2\nu-2}\right), \tag{47}$$

where in the first (second) case all sign combinations with an even (odd) number of minus signs occur. This is reflected by the indices "e" and "o" in our result for the partition function,

$$\begin{aligned} Z(a,b) &= [2\sinh(2a)]^{mn/2}\left[\sum_{\mathrm{e}}\exp\left(\frac{m}{2}\sum_{\nu=1}^{n}(\pm)\gamma_{2\nu-1}\right)+\sum_{\mathrm{o}}\exp\left(\frac{m}{2}\sum_{\nu=1}^{n}(\pm)\gamma_{2\nu-2}\right)\right] \\ &= \frac{1}{2}[2\sinh(2a)]^{mn/2} \\ &\quad\times\left\{\prod_{k=1}^{n}\left[2\cosh\left(\frac{m}{2}\gamma_{2k-1}\right)\right]+\prod_{k=1}^{n}\left[2\sinh\left(\frac{m}{2}\gamma_{2k-1}\right)\right]+\prod_{k=1}^{n}\left[2\cosh\left(\frac{m}{2}\gamma_{2k-2}\right)\right]-\prod_{k=1}^{n}\left[2\sinh\left(\frac{m}{2}\gamma_{2k-2}\right)\right]\right\}. \end{aligned} \tag{48}$$

The last term within the braces has a sign differing from that in [4]. This is due to our different sign convention for γ_0. The eigenvalues of T are of course obtained by multiplying Eqs. (46) and (47) with the trivial factor $[2\sinh(2a)]^{n/2}$.

The results for the eigenvalues of T and the partition function are the starting point for the analysis of the thermodynamic properties of the two-dimensional Ising model, the most interesting case being the thermodynamic limit $m,n\to\infty$. Such analyses can now proceed as usual (see, e.g., [1, 4, 8]) and will not be repeated here.

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